

All-loop group-theory constraints for color-ordered $SU(N)$ gauge-theory amplitudes

Stephen G. Naculich¹

*Department of Physics
Bowdoin College
Brunswick, ME 04011, USA*

`naculich@bowdoin.edu`

Abstract

We derive constraints on the color-ordered amplitudes of the L -loop four-point function in $SU(N)$ gauge theories that arise solely from the structure of the gauge group. These constraints generalize well-known group theory relations, such as $U(1)$ decoupling identities, to all loop orders.

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1 Introduction

An exciting recent development in the study of perturbative amplitudes is the discovery of color-kinematic duality of gauge theory amplitudes at both tree and loop level [1, 2]. This duality implies the existence of constraints on tree-level color-ordered amplitudes, which were proven in refs. [3–6]. The BCJ conjecture was also verified through three loops for the $\mathcal{N} = 4$ supersymmetric Yang-Mills four- [2, 7] and five-point [8, 9] amplitudes. (See reviews in refs. [7, 10], which also contain references to related work on the subject.)

The BCJ constraints on tree-level color-ordered amplitudes hold in addition to various well-known $SU(N)$ group theory relations, such as the $U(1)$ decoupling or dual Ward identity [11, 12] and the Kleiss-Kuijf relations [13]. Group-theory relations also hold for one-loop [14, 15] and two-loop [16] color-ordered amplitudes. They can be elegantly derived by using an alternative color decomposition of the amplitude [17, 18].

The purpose of this note is extend the $SU(N)$ group theory relations for four-point amplitudes to all loops. We develop a recursive procedure to derive constraints satisfied by any L -loop diagram (containing only adjoint fields) obtained by attaching a rung between two external legs of an $(L - 1)$ -loop diagram. We assume that the most general L -loop color factor can be obtained from this subset using Jacobi relations, an assumption that has been proven through $L = 4$. Using this method, we find four independent group-theory constraints for color-ordered four-point amplitudes at each loop level (except for $L = 0$ and $L = 1$, where there are one and three constraints respectively).

The color-ordered amplitudes of a gauge theory are the coefficients of the full amplitude in a basis using traces of generators in the fundamental representation of the gauge group. Color-ordered amplitudes have the advantage of being individually gauge-invariant. Four-point amplitudes of $SU(N)$ gauge theories can be expressed in terms of single and double traces [14]

$$\begin{aligned} T_1 &= \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}) + \text{Tr}(T^{a_1} T^{a_4} T^{a_3} T^{a_2}), & T_4 &= \text{Tr}(T^{a_1} T^{a_3}) \text{Tr}(T^{a_2} T^{a_4}), \\ T_2 &= \text{Tr}(T^{a_1} T^{a_2} T^{a_4} T^{a_3}) + \text{Tr}(T^{a_1} T^{a_3} T^{a_4} T^{a_2}), & T_5 &= \text{Tr}(T^{a_1} T^{a_4}) \text{Tr}(T^{a_2} T^{a_3}), \\ T_3 &= \text{Tr}(T^{a_1} T^{a_4} T^{a_2} T^{a_3}) + \text{Tr}(T^{a_1} T^{a_3} T^{a_2} T^{a_4}), & T_6 &= \text{Tr}(T^{a_1} T^{a_2}) \text{Tr}(T^{a_3} T^{a_4}). \end{aligned} \quad (1.1)$$

All other possible trace terms vanish in $SU(N)$ since $\text{Tr}(T^a) = 0$. The color-ordered amplitudes can be further decomposed [19] in powers of N as

$$\mathcal{A}^{(L)} = \sum_{\lambda=1}^3 \left(\sum_{k=0}^{\lfloor \frac{L}{2} \rfloor} N^{L-2k} A_{\lambda}^{(L,2k)} \right) T_{\lambda} + \sum_{\lambda=4}^6 \left(\sum_{k=0}^{\lfloor \frac{L-1}{2} \rfloor} N^{L-2k-1} A_{\lambda}^{(L,2k+1)} \right) T_{\lambda} \quad (1.2)$$

where $A_{\lambda}^{(L,0)}$ are leading-order-in- N (planar) amplitudes, and $A_{\lambda}^{(L,k)}$, $k = 1, \dots, L$, are subleading-order, yielding $(3L + 3)$ color-ordered amplitudes at L loops.

Alternatively, amplitudes may be decomposed into a basis of color factors [17, 18]. It is in such a basis that color-kinematic duality is manifest [1, 2]. The number of linearly-independent L -loop color factors, however, is less than the number of elements of the L -loop trace basis, implying the

existence of constraints among $A_\lambda^{(L,k)}$. In this note we show that, for even L , the color-ordered amplitudes must satisfy

$$6 \sum_{\lambda=1}^3 A_\lambda^{(L,L-2)} - \sum_{\lambda=4}^6 A_\lambda^{(L,L-1)} = 0, \quad (1.3)$$

$$A_{\lambda+3}^{(L,L-1)} + A_\lambda^{(L,L)} = \text{independent of } \lambda, \quad (1.4)$$

$$\sum_{\lambda=1}^3 A_\lambda^{(L,L)} = 0, \quad (1.5)$$

while for odd L , the relations are

$$6 \sum_{\lambda=1}^3 A_\lambda^{(L,L-3)} - \sum_{\lambda=4}^6 A_\lambda^{(L,L-2)} + 2 \sum_{\lambda=1}^3 A_\lambda^{(L,L-1)} = 0, \quad (1.6)$$

$$6 \sum_{\lambda=1}^3 A_\lambda^{(L,L-1)} - \sum_{\lambda=4}^6 A_\lambda^{(L,L)} = 0, \quad (1.7)$$

$$A_\lambda^{(L,L)} = \text{independent of } \lambda. \quad (1.8)$$

These constraints generalize known group theory relations at tree-level [11,12], one loop [14], and two loops [16] to all loop orders. In particular, we note that eqs. (1.3), (1.5), (1.7), and (1.8) can alternatively be derived by expanding the amplitude in a $U(N)$ trace basis and requiring that any amplitude containing one or more gauge bosons in the $U(1)$ subgroup vanish. Such $U(1)$ decoupling arguments, however, cannot be used to obtain eqs. (1.4) and (1.6).

Since the space of L -loop color factors is by construction at least $(3L-1)$ -dimensional (for $L \geq 2$), eqs. (1.3)-(1.8) are the maximal set of constraints on color-ordered amplitudes that follow from $SU(N)$ group theory alone.² It is interesting that these constraints only involve the three or four most-subleading-in- $1/N$ color-ordered amplitudes at a given loop order; other amplitudes are not constrained at all by group theory. Of course, color-kinematic duality implies further relations among the amplitudes [1,2]. Other recent work on constraints among loop-level amplitudes includes refs. [20–22].

In sec. 2, we describe the relation between color and trace bases, and how to use this to derive constraints among color-ordered amplitudes. In sec. 3, we apply this to four-point amplitudes through two loops, and then develop and solve all-loop-order recursion relations yielding constraints for four-point color-ordered amplitudes. In the appendix, we provide details about the three- and four-loop cases.

2 Color and trace bases

In this section, we schematically outline the approach we use to obtain constraints among color-ordered amplitudes. This approach was used in ref. [23] for tree-level and one-loop five-point

²If our recursive procedure together with the Jacobi relations do not yield the entire space of L -loop color factors, then some of these constraints could be violated for $L > 4$, though we think this unlikely.

amplitudes.

The n -point amplitude in a gauge theory containing only fields in the adjoint representation of $SU(N)$ (such as pure Yang-Mills or supersymmetric Yang-Mills theory) can be written in a loop expansion, with the L -loop contribution given by a sum of L -loop Feynman diagrams. Suppressing n and L , as well as all momentum and polarization dependence, we can express the L -loop amplitude in the “parent-graph” decomposition [24]

$$\mathcal{A} = \sum_i a_i c_i \quad (2.1)$$

where $\{c_i\}$ represents a complete set of color factors of L -loop n -point diagrams built from cubic vertices with a factor of the $SU(N)$ structure constants \tilde{f}^{abc} at each vertex. Contributions from Feynman diagrams containing quartic vertices with factors of $\tilde{f}^{abe}\tilde{f}^{cde}$, $\tilde{f}^{ace}\tilde{f}^{bde}$, and $\tilde{f}^{ade}\tilde{f}^{bce}$ can be parceled out among other diagrams containing only cubic vertices. The set of color factors may be overcomplete, in which case they satisfy relations of the form

$$\sum_i \ell_i c_i = 0. \quad (2.2)$$

In fact, it is often necessary to use an overcomplete basis to make color-kinematic duality manifest [1, 8]. Although the amplitude (2.1) is gauge invariant, the individual terms in the sum may not be. Any gauge-dependent pieces of the form $a_i = \ell_i f$ (where f is independent of i) will cancel out due to eq. (2.2).

The L -loop amplitude may alternatively be expressed in terms of a trace basis $\{t_\lambda\}$ as

$$\mathcal{A} = \sum_\lambda A_\lambda t_\lambda \quad (2.3)$$

where A_λ are gauge-invariant color-ordered amplitudes. One can convert the amplitude (2.1) into the trace basis by writing

$$\tilde{f}^{abc} = i\sqrt{2}f^{abc} = \text{Tr}([T^a, T^b]T^c) \quad (2.4)$$

and using the $SU(N)$ identities

$$\begin{aligned} \text{Tr}(PT^a)\text{Tr}(QT^a) &= \text{Tr}(PQ) - \frac{1}{N}\text{Tr}(P)\text{Tr}(Q) \\ \text{Tr}(PT^aQT^a) &= \text{Tr}(P)\text{Tr}(Q) - \frac{1}{N}\text{Tr}(PQ) \end{aligned} \quad (2.5)$$

to express the color factor c_i as a linear combination of traces

$$c_i = \sum_\lambda M_{i\lambda} t_\lambda. \quad (2.6)$$

The color-ordered amplitudes are then given by

$$A_\lambda = \sum_i a_i M_{i\lambda}. \quad (2.7)$$

Any constraints (2.2) among the color factors correspond to left null eigenvectors of the transformation matrix

$$\sum_i \ell_i M_{i\lambda} = 0. \quad (2.8)$$

The transformation matrix will also have a set of right null eigenvectors

$$\sum_\lambda M_{i\lambda} r_\lambda = 0. \quad (2.9)$$

Each right null eigenvector implies a constraint

$$\sum_\lambda A_\lambda r_\lambda = 0 \quad (2.10)$$

on the color-ordered amplitudes.

3 Constraints on color-ordered four-point amplitudes

In eq. (1.2), we decomposed the L -loop four-point amplitude in terms of the six-dimensional trace basis $\{T_\lambda\}$ defined in eq. (1.1). The $1/N$ expansion suggests enlarging the trace basis to the $(3L+3)$ -dimensional basis $\{t_\lambda^{(L)}\}$:

$$\begin{aligned} t_{1+6k}^{(L)} &= N^{L-2k} T_1, & t_{4+6k}^{(L)} &= N^{L-2k-1} T_4, \\ t_{2+6k}^{(L)} &= N^{L-2k} T_2, & t_{5+6k}^{(L)} &= N^{L-2k-1} T_5, \\ t_{3+6k}^{(L)} &= N^{L-2k} T_3, & t_{6+6k}^{(L)} &= N^{L-2k-1} T_6, \end{aligned} \quad (3.1)$$

in terms of which eq. (1.2) becomes

$$\mathcal{A}^{(L)} = \sum_{\lambda=1}^{3L+3} A_\lambda^{(L)} t_\lambda^{(L)}, \quad \text{where} \quad A_{\lambda+6k}^{(L)} = \begin{cases} A_\lambda^{(L,2k)}, & \lambda = 1, 2, 3, \\ A_\lambda^{(L,2k+1)}, & \lambda = 4, 5, 6. \end{cases} \quad (3.2)$$

The decomposition (2.6) of color factors c_i into the trace basis $\{t_\lambda^{(L)}\}$ shows that the number of independent L -loop color factors cannot exceed $3L+3$. The dimension of the space of color factors is actually less than this, being 2-dimensional at tree level, 3-dimensional at one loop, and $(3L-1)$ -dimensional for $L \geq 2$ (only proven for $L \leq 4$). As we will illustrate below, this implies the existence of right null eigenvectors (2.9) of the transformation matrix $M_{i\lambda}^{(L)}$ and corresponding constraints (2.10) among the color-ordered amplitudes $A_\lambda^{(L)}$.

At tree level, the space of color factors is spanned by the t -channel exchange diagram

$$C_{st}^{(0)} = \tilde{f}^{a_1 a_4 b} \tilde{f}^{a_3 a_2 b} = t_1^{(0)} - t_3^{(0)} \quad (3.3)$$

and the corresponding s -channel exchange diagram

$$C_{ts}^{(0)} = \tilde{f}^{a_1 a_2 b} \tilde{f}^{a_3 a_4 b} = t_1^{(0)} - t_2^{(0)}. \quad (3.4)$$

The u -channel diagram is related to these by the Jacobi identity. With $\{c_1, c_2\} = \{C_{st}^{(0)}, C_{ts}^{(0)}\}$, the transformation matrix (2.6) and its right null eigenvector (2.9) are

$$M_{i\lambda}^{(0)} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \quad r^{(0)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad (3.5)$$

which implies the U(1) decoupling identity among color-ordered tree amplitudes [11, 12]

$$A_1^{(0)} + A_2^{(0)} + A_3^{(0)} = 0. \quad (3.6)$$

This is eq. (1.5) for $L = 0$.

The color factor of the one-loop box diagram

$$C_{st}^{(1)} = C_{ts}^{(1)} = \tilde{f}^{ea_1b} \tilde{f}^{ba_2c} \tilde{f}^{ca_3d} \tilde{f}^{da_4e} = t_1^{(1)} + 2(t_4^{(1)} + t_5^{(1)} + t_6^{(1)}) \quad (3.7)$$

and its independent permutations $C_{us}^{(1)}$ and $C_{tu}^{(1)}$ span the space of one-loop color factors, giving

$$M_{i\lambda}^{(1)} = \begin{pmatrix} 1 & 0 & 0 & 2 & 2 & 2 \\ 0 & 1 & 0 & 2 & 2 & 2 \\ 0 & 0 & 1 & 2 & 2 & 2 \end{pmatrix}. \quad (3.8)$$

Alternatively, we can choose³ for our basis $NC_{st}^{(0)}$ and $NC_{ts}^{(0)}$, together with $C_{st}^{(1)}$, to give

$$M_{i\lambda}^{(1)} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 2 \end{pmatrix}. \quad (3.9)$$

In either case, the transformation matrix has three independent right null eigenvectors

$$r^{(1)} = \begin{pmatrix} 6u \\ -u \end{pmatrix}, \quad \begin{pmatrix} 0 \\ x \end{pmatrix}, \quad \begin{pmatrix} 0 \\ y \end{pmatrix}, \quad \text{where} \quad u \equiv \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad x \equiv \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad y \equiv \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad (3.10)$$

implying three relations among the one-loop color-ordered amplitudes [14]

$$A_4^{(1)} = A_5^{(1)} = A_6^{(1)} = 2(A_1^{(1)} + A_2^{(1)} + A_3^{(1)}). \quad (3.11)$$

These are eqs. (1.7) and (1.8) for $L = 1$.

At two loops, the ladder and non-planar diagrams⁴ yield the color factors

$$C_{st}^{(2L)} = \tilde{f}^{ea_1b} \tilde{f}^{ba_2c} \tilde{f}^{cga_3d} \tilde{f}^{dfe} \tilde{f}^{ga_3h} \tilde{f}^{ha_4f} = t_1^{(2)} + 6t_6^{(2)} + 2t_7^{(2)} + 2t_8^{(2)} - 4t_9^{(2)}, \quad (3.12)$$

$$C_{st}^{(2NP)} = \tilde{f}^{ea_1b} \tilde{f}^{ba_2c} \tilde{f}^{cga_3d} \tilde{f}^{hfe} \tilde{f}^{ga_3h} \tilde{f}^{da_4f} = -2t_4^{(2)} - 2t_5^{(2)} + 4t_6^{(2)} + 2t_7^{(2)} + 2t_8^{(2)} - 4t_9^{(2)}. \quad (3.13)$$

³This makes sense since we can use the Jacobi identity to replace the one-loop box diagram with another box diagram with permuted legs plus a tree diagram with one of the vertices replaced by a triangle diagram. The latter is proportional to a tree diagram since $\tilde{f}^{da_1b} \tilde{f}^{ba_2c} \tilde{f}^{ca_3d} = N \tilde{f}^{a_1a_2a_3}$.

⁴It can be easily shown that any other two-loop diagram is related to these ones by Jacobi relations.

The non-planar color factors can be expressed in terms of the planar ones,

$$3C_{st}^{(2NP)} = C_{st}^{(2L)} - C_{ts}^{(2L)} - C_{us}^{(2L)} + C_{su}^{(2L)}, \quad (3.14)$$

and a linear relation exists among the planar color factor and its permutations,

$$0 = C_{st}^{(2L)} - C_{ts}^{(2L)} + C_{us}^{(2L)} - C_{su}^{(2L)} + C_{tu}^{(2L)} - C_{ut}^{(2L)}. \quad (3.15)$$

We could therefore choose five of the six permutations of the ladder diagram to span the space of two-loop color factors; alternatively, we can use $N^2 C_{st}^{(0)}$, $N^2 C_{ts}^{(0)}$, and $NC_{st}^{(1)}$, together with $C_{st}^{(2L)}$ and $C_{ts}^{(2L)}$, to obtain

$$M_{i\lambda}^{(2)} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 6 & 2 & 2 & -4 \\ 1 & 0 & 0 & 0 & 6 & 0 & 2 & -4 & 2 \end{pmatrix}. \quad (3.16)$$

The two-loop transformation matrix has four independent right null eigenvectors

$$r^{(2)} = \begin{pmatrix} 6u \\ -u \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ x \\ x \end{pmatrix}, \quad \begin{pmatrix} 0 \\ y \\ y \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} \quad (3.17)$$

implying four two-loop group-theory relations [16]

$$\begin{aligned} 0 &= A_4^{(2)} + A_7^{(2)} - 2(A_1^{(2)} + A_2^{(2)} + A_3^{(2)}) \\ 0 &= A_5^{(2)} + A_8^{(2)} - 2(A_1^{(2)} + A_2^{(2)} + A_3^{(2)}) \\ 0 &= A_6^{(2)} + A_9^{(2)} - 2(A_1^{(2)} + A_2^{(2)} + A_3^{(2)}) \\ 0 &= A_7^{(2)} + A_8^{(2)} + A_9^{(2)} \end{aligned} \quad (3.18)$$

equivalent to eqs. (1.3)-(1.5) for $L = 2$.

We now employ a recursive procedure to obtain null eigenvectors for higher-loop color factors. An $(L+1)$ -loop diagram may be obtained from an L -loop diagram by attaching a rung between two of its external legs, i and j . This corresponds to contracting its color factor with $\tilde{f}^{a_i a'_i b} \tilde{f}^{b a'_j a_j}$. Note that if i and j are not adjacent, this will convert a planar diagram into a nonplanar diagram. First consider the effect of this procedure [25] on the trace basis (1.1)

$$T_\lambda \longrightarrow \sum_{\kappa=1}^6 G_{\lambda\kappa} T_\kappa, \quad \text{where} \quad G = \begin{pmatrix} NA & B \\ C & ND \end{pmatrix} \quad (3.19)$$

with

$$\begin{aligned} A &= \begin{pmatrix} e_{12} + e_{14} & 0 & 0 \\ 0 & e_{12} + e_{13} & 0 \\ 0 & 0 & e_{13} + e_{14} \end{pmatrix}, & B &= \begin{pmatrix} 0 & 2e_{14} - 2e_{13} & 2e_{12} - 2e_{13} \\ 2e_{13} - 2e_{14} & 0 & 2e_{12} - 2e_{14} \\ 2e_{13} - 2e_{12} & 2e_{14} - 2e_{12} & 0 \end{pmatrix}, \\ C &= \begin{pmatrix} 0 & e_{12} - e_{14} & e_{14} - e_{12} \\ e_{12} - e_{13} & 0 & e_{13} - e_{12} \\ e_{14} - e_{13} & e_{13} - e_{14} & 0 \end{pmatrix}, & D &= \begin{pmatrix} 2e_{13} & 0 & 0 \\ 0 & 2e_{14} & 0 \\ 0 & 0 & 2e_{12} \end{pmatrix}, \end{aligned} \quad (3.20)$$

where the coefficient of e_{1j} corresponds to connecting legs 1 and j . On the expanded basis (3.1), the same procedure yields

$$t_\lambda^{(L)} \rightarrow \sum_{\kappa=1}^{3L+6} g_{\lambda\kappa} t_\kappa^{(L+1)} \quad (3.21)$$

where g is the $(3L+3) \times (3L+6)$ matrix

$$g = \begin{pmatrix} A & B & 0 & 0 & 0 & \dots \\ 0 & D & C & 0 & 0 & \dots \\ 0 & 0 & A & B & 0 & \dots \\ 0 & 0 & 0 & D & C & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \quad (3.22)$$

Next, given some L -loop diagram with color factor $c_i^{(L)}$, we can connect two of its external legs with a rung to obtain an $(L+1)$ -loop diagram with color factor

$$c_i^{(L+1)} = \sum_{\kappa=1}^{3L+6} M_{i\kappa}^{(L+1)} t_\kappa^{(L+1)} \quad (3.23)$$

where

$$M_{i\kappa}^{(L+1)} = \sum_{\lambda=1}^{3L+3} M_{i\lambda}^{(L)} g_{\lambda\kappa}, \quad \text{with} \quad c_i^{(L)} = \sum_{\lambda=1}^{3L+3} M_{i\lambda}^{(L)} t_\lambda^{(L)}. \quad (3.24)$$

Now, suppose we possess a complete set of L -loop color factors $\{c_i^{(L)}\}$ and a maximal set of right null eigenvectors $\{r_\lambda^{(L)}\}$:

$$\sum_{\lambda=1}^{3L+3} M_{i\lambda}^{(L)} r_\lambda^{(L)} = 0. \quad (3.25)$$

Then the color factors of all $(L+1)$ -loop diagrams obtained by connecting two external legs of any L -loop diagram will have a right null eigenvector

$$\sum_{\kappa=1}^{3L+6} M_{i\kappa}^{(L+1)} r_\kappa^{(L+1)} = 0 \quad (3.26)$$

provided that $r_\kappa^{(L+1)}$ satisfies

$$\sum_{\kappa=1}^{3L+6} g_{\lambda\kappa} r_\kappa^{(L+1)} = \text{linear combination of } \{r_\lambda^{(L)}\}. \quad (3.27)$$

We can now solve eq. (3.27) recursively, beginning with the set of $L=2$ right null eigenvectors (3.17), the first case with four independent eigenvectors. The maximal set of right null

eigenvectors satisfying eq. (3.27) is

$$\{r^{(2\ell+1)}\} = \begin{pmatrix} \vdots \\ 0 \\ 6u \\ -u \\ 2u \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ 6u \\ -u \end{pmatrix}, \quad \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ x \end{pmatrix}, \quad \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ 0 \\ y \end{pmatrix}, \quad \{r^{(2\ell)}\} = \begin{pmatrix} \vdots \\ 0 \\ 6u \\ -u \\ 0 \end{pmatrix}, \quad \begin{pmatrix} \vdots \\ 0 \\ 0 \\ x \\ x \end{pmatrix}, \quad \begin{pmatrix} \vdots \\ 0 \\ 0 \\ y \\ y \end{pmatrix}, \quad \begin{pmatrix} \vdots \\ 0 \\ 0 \\ 0 \\ u \end{pmatrix}. \quad (3.28)$$

The constraints on color-ordered amplitudes

$$\sum_{\lambda} A_{\lambda}^{(L)} r_{\lambda}^{(L)} = 0 \quad (3.29)$$

that follow from the set of right null eigenvectors (3.28) can be written in terms of eq. (3.2) to yield the constraints (1.3)-(1.8) given in the introduction.

Since there are generally four⁵ linearly-independent null eigenvectors in a $(3L+3)$ -dimensional trace space, the space of L -loop color factors satisfying eq. (3.25) is generally $(3L-1)$ -dimensional.⁶ Since there are no further independent solutions of eq. (3.27), we have shown that the full space of L -loop color factors is *at least* $(3L-1)$ -dimensional.

We have not strictly shown that eq. (3.28) are null eigenvectors for *any* possible color factor associated with an L -loop diagram, but rather only for those that can be obtained from an $(L-1)$ -loop diagram by attaching a rung between two external legs. It is therefore conceivable (but we think unlikely) that the space of *all* L -loop color factors could be greater than $(3L-1)$ -dimensional. However, for $L=3$ and $L=4$, it has been shown [24] that, despite the fact that many diagrams cannot be obtained by attaching a rung to the external legs of lower-loop diagrams, all color factors can be related to these using Jacobi relations (see the appendix for further discussion of $L=3$ and $L=4$). It would be nice to have a proof of this for all L , however.

4 Conclusions

In this note, we have extended known group theory identities for four-point color-ordered amplitudes in $SU(N)$ gauge theories to all loop orders. We have shown that color-ordered amplitude generally must satisfy four independent relations at each loop order (except for $L=0$ and $L=1$, where there are one and three constraints respectively). This was achieved via a recursive procedure that derives the constraints on L -loop color factors generated by attaching a rung between two external legs of an $(L-1)$ -loop color factor. Assuming that all L -loop color factors are linear combinations of those just described (i.e., via Jacobi relations), then the constraints derived apply to all L -loop color-ordered amplitudes. Although this has been established through four loops, it would clearly be desirable to have an all-orders proof of this assumption.

⁵One for $L=0$ and three for $L=1$.

⁶Two-dimensional for $L=0$ and three-dimensional for $L=1$.

The recursive method employed in this note can also be extended to n -point functions with $n > 4$ to yield constraints on the color-ordered amplitudes beyond those already known at tree- [13] and one-loop [14, 15] level, although the size of the color basis grows quickly with n .

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A Appendix

In appendix B of ref. [24], bases for the space of all three- and four-loop color factors were identified. In this appendix, we explicitly check that the right null eigenvectors of these spaces coincide with our recursive solution (3.28), and therefore that all three- and four-loop color-ordered amplitudes indeed satisfy the group theory constraints eqs. (1.3)-(1.8).

The basis for three-loop color factors can be chosen as $N^3 C_{st}^{(0)}$, $N^3 C_{ts}^{(0)}$, $N^2 C_{st}^{(1)}$, $N C_{st}^{(2L)}$, and $N C_{ts}^{(2L)}$, plus the color factor for the three-loop ladder diagram

$$C_{st}^{(3L)} = t_1^{(3)} + 14t_6^{(3)} + 2t_7^{(3)} + 2t_8^{(3)} + 8t_{10}^{(3)} + 8t_{11}^{(3)} + 8t_{12}^{(3)} \quad (\text{A.1})$$

and two of its permutations,⁷ $C_{ts}^{(3L)}$ and $C_{us}^{(3L)}$, yielding the transformation matrix

$$M_{i\lambda}^{(3)} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 6 & 2 & 2 & -4 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 6 & 0 & 2 & -4 & 2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 14 & 2 & 2 & 0 & 8 & 8 & 8 \\ 1 & 0 & 0 & 0 & 14 & 0 & 2 & 0 & 2 & 8 & 8 & 8 \\ 0 & 1 & 0 & 14 & 0 & 0 & 0 & 2 & 2 & 8 & 8 & 8 \end{pmatrix}. \quad (\text{A.2})$$

The four independent right null eigenvectors of this matrix

$$r^{(3)} = \begin{pmatrix} 6u \\ -u \\ 2u \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 6u \\ -u \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ x \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ y \end{pmatrix} \quad (\text{A.3})$$

agree with those in eq. (3.28), and imply the four constraints among the color-ordered amplitudes given by eqs. (1.6)-(1.8) with $L = 3$.

⁷ Only $C_{st}^{(3L)}$ is used in ref. [24], but the authors also include $N C_{st}^{(0)}$ and $N C_{ts}^{(0)}$ in their basis, which in our approach are independent of $N^3 C_{st}^{(0)}$ and $N^3 C_{ts}^{(0)}$.

The four-loop color basis can be chosen as (N times) the three-loop basis plus three color factors from the four-loop ladder diagram and two⁸ permutations, $C_{st}^{(4L)}$, $C_{ts}^{(4L)}$, and $C_{us}^{(4L)}$, yielding

$$M_{i\lambda}^{(4)} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 2 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 6 & 2 & 2 & -4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 6 & 0 & 2 & -4 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 14 & 2 & 2 & 0 & 8 & 8 & 8 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 14 & 0 & 2 & 0 & 2 & 8 & 8 & 8 & 0 & 0 & 0 \\ 0 & 1 & 0 & 14 & 0 & 0 & 0 & 2 & 2 & 8 & 8 & 8 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 30 & 2 & 2 & 0 & 0 & 0 & 24 & 8 & 8 & 16 \\ 1 & 0 & 0 & 0 & 30 & 0 & 2 & 0 & 2 & 0 & 24 & 0 & 8 & 16 & 8 \\ 0 & 1 & 0 & 30 & 0 & 0 & 0 & 2 & 2 & 24 & 0 & 0 & 16 & 8 & 8 \end{pmatrix}. \quad (\text{A.4})$$

The four independent right null eigenvectors of this matrix

$$r^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 6u \\ -u \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ x \\ x \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ y \\ y \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ u \end{pmatrix}, \quad (\text{A.5})$$

agree with those in eq. (3.28). The right null eigenvalues imply the four relations among color-ordered amplitudes given by eqs. (1.3)-(1.5) for $L = 4$.

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⁸ Only $C_{st}^{(4L)}$ and $C_{ts}^{(4L)}$ are used in ref. [24], but the authors also include $NC_{st}^{(1)}$, which in our approach counts as independent from $N^3 C_{st}^{(1)}$.

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